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# On the Robustness of Hysteretic Second-Order Systems with PID: *i*ISS approach

Ruiyue Ouyang, Bayu Jayawardhana, Vincent Andrieu

**Abstract**—In this paper, we study the robustness property of a second-order linear plant controlled by a proportional, integral and derivative (PID) controller with a hysteretic actuator. The hysteretic actuator is modeled by a Duhem model that exhibits clockwise (CW) input-output (I/O) dynamics (such as the Dahl model, LuGre model and Maxwell-Slip model, which describe hysteresis phenomena in mechanical friction). Based on our main result, we provide sufficient conditions on the controller gains that depend on the state of the hysteresis is globally attractive. The robustness of the closed-loop system with respect to the measurement noise is also given, using the *integral* input-to-state stability (*i*ISS) concept. The applicability of the robustness analysis in the design of state observer for such systems is shown in the simulation.

## I. INTRODUCTION

Hysteresis is a common nonlinear phenomenon that exists in many physical systems. The input-output (I/O) behavior of a hysteresis phenomenon can be characterized into counterclockwise (CCW) [1], clockwise (CW) [12], or even more complex I/O map (such as, butterfly map [3]). For example, in the ferromagnetic material, the hysteresis phenomena exhibits CCW I/O behavior between the electrical field and the magnetization [4], [9], [11]; in a friction-induced mechanical system, the hysteresis exhibits CW I/O behavior between the friction force and the displacement [13], [5], and in a two-bar linkage system, the hysteresis exhibits butterfly I/O behavior between the vertical force on the joint and the horizontal displacement [6].

In this paper, we study the stability property of a second-order linear plant controlled by a proportional, integral and derivative (PID) controller with a hysteretic actuator as shown in Figure 1. For the hysteretic actuator part, we consider the class of the Duhem hysteresis operator with CW I/O behavior, which can describe hysteretic friction phenomena. It has been shown in [13] that the classical friction hysteresis models such as Dahl model, LuGre model and Maxwell-Slip model can be recast into the Duhem model, which has CW I/O behavior. An example to this problem is the position control of a (micro-)vehicle where we manipulate the rotation of the wheels in order to exert forces to the vehicle's body via friction forces.

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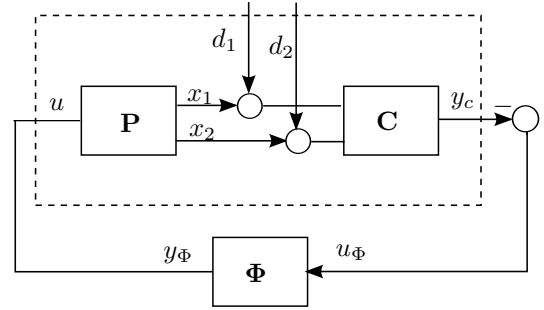


Fig. 1. Negative feedback interconnection with a linear system  $P$ , a controller  $C$  and a Duhem hysteresis operator  $\Phi$ .

In our previous results in [15], we show that for a certain class of Duhem hysteresis operator  $\Phi : AC(\mathbb{R}_+) \times \mathbb{R} \rightarrow AC(\mathbb{R}_+)$ , where  $AC$  is the class of absolutely continuous functions, we can construct a function  $H_{\circ} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  which satisfies

$$\frac{dH_{\circ}(y_{\Phi}(t), u_{\Phi}(t))}{dt} \leq \dot{u}_{\Phi}(t)y_{\Phi}(t),$$

where  $u_{\Phi} \in AC(\mathbb{R}_+)$  and  $y_{\Phi} := \Phi(u_{\Phi}, y_{\Phi_0})$ ,  $y_{\Phi_0} \in \mathbb{R}$ . This inequality immediately implies that such Duhem hysteresis operator is dissipative with respect to the supply rate  $\dot{u}_{\Phi}(t)y_{\Phi}(t)$  and has CW I/O dynamics which will be defined explicitly in Section II. The symbol  $\circ$  in  $H_{\circ}$  indicates the clockwise behavior of  $\Phi$ . A related result on the CCW Duhem hysteresis operator is discussed in [8].

In our main result, the knowledge of  $H_{\circ}$  facilitates the stability analysis of the closed-loop systems. In particular, we give sufficient conditions on the plant and the controller such that the origin on the plant and the state of the hysteresis is globally attractive. The analysis provides a construction method to determine the controller gains based only on the plant parameters. The robustness of the closed-loop system with respect to the measurement noise ( $d_1, d_2$ ) is also given, using the *integral* input-to-state stability (*i*ISS) concept. This robustness analysis can be useful in the design of state observer for such systems through the application of small-gain principle.

This paper is organized as follows. In Section II we give some notations and definitions which are used throughout the paper. In Section III we show the stability analysis for a second order system which is feedback interconnected with a CW Duhem hysteresis and a PID controller. In addition, the robustness of the closed-loop system are discussed. The simulation results are shown in Section IV.

## II. PRELIMINARIES

Let  $C^1(\mathbb{R}_+)$  be the space of continuously differentiable functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ . Denote  $AC(\mathbb{R}_+)$  the space of absolutely continuous functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ . Define  $\frac{dz(t)}{dt} := \lim_{h \searrow 0} \frac{z(t+h) - z(t)}{h}$ .

### A. Counterclockwise and Clockwise dynamics

The following two definitions of the CCW and CW dynamics are based on the work by Angeli [1] and Padthe [12].

**Definition 2.1:** [1] A (nonlinear) map  $G : AC(\mathbb{R}_+) \rightarrow AC(\mathbb{R}_+)$  is *counterclockwise* (CCW) if for every  $u \in AC(\mathbb{R}_+)$  with the corresponding output map  $y := G(u)$ , the following inequality holds

$$\liminf_{T \rightarrow \infty} \int_0^T \dot{y}(t)u(t)dt > -\infty. \quad (1)$$

**Definition 2.2:** [12] A (nonlinear) map  $G : AC(\mathbb{R}_+) \rightarrow AC(\mathbb{R}_+)$  is *clockwise* (CW) if for every  $u \in AC(\mathbb{R}_+)$  with the corresponding output map  $y := G(u)$ , the following inequality holds

$$\liminf_{T \rightarrow \infty} \int_0^T y(t)\dot{u}(t)dt > -\infty. \quad (2)$$

For a nonlinear operator  $G$ , inequality (1) holds if there exists a function  $H_\circ : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that for every input signal  $u \in AC(\mathbb{R}_+)$ , the inequality

$$\frac{dH_\circ(y(t), u(t))}{dt} \leq \dot{y}(t)u(t), \quad (3)$$

holds for almost every  $t$  where the output signal  $y := G(u)$ .

Consequently, for a nonlinear operator  $G$ , inequality (2) holds if there exists a function  $H_\circ : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that for every input signal  $u \in AC(\mathbb{R}_+)$ , the inequality

$$\frac{dH_\circ(y(t), u(t))}{dt} \leq y(t)\dot{u}(t), \quad (4)$$

holds for a.e.  $t$  where the output signal  $y := G(u)$ .

### B. Duhem hysteresis operator

The Duhem operator  $\Phi : AC(\mathbb{R}_+) \times \mathbb{R} \rightarrow AC(\mathbb{R}_+)$ ,  $u_\Phi \mapsto \Phi(u_\Phi, y_{\Phi_0}) =: y_\Phi$  is described by

$$\begin{aligned} \dot{y}_\Phi(t) &= f_1(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi+}(t) + f_2(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi-}(t), \\ y_\Phi(0) &= y_{\Phi_0}, \end{aligned} \quad (5)$$

where  $\dot{u}_{\Phi+}(t) := \max\{0, \dot{u}_\Phi(t)\}$ ,  $\dot{u}_{\Phi-}(t) := \min\{0, \dot{u}_\Phi(t)\}$  (see also, [10], [11], [20]). The functions  $f_1$  and  $f_2$  in (5) are assumed to be  $C^1$ .

The existence of solutions to (5) has been reviewed in [10]. In particular, if for every  $\xi \in \mathbb{R}$ ,  $f_1$  and  $f_2$  satisfy

$$\begin{aligned} (\nu_1 - \nu_2)[f_1(\nu_1, \xi) - f_1(\nu_2, \xi)] &\leq \lambda_1(\xi)(\nu_1 - \nu_2)^2, \\ (\nu_1 - \nu_2)[f_2(\nu_1, \xi) - f_2(\nu_2, \xi)] &\geq -\lambda_2(\xi)(\nu_1 - \nu_2)^2, \end{aligned} \quad (6)$$

for all  $\nu_1, \nu_2 \in \mathbb{R}$ , where  $\lambda_1$  and  $\lambda_2$  are nonnegative, then the solution to (5) exist and  $\Phi$  maps  $AC(\mathbb{R}_+) \times \mathbb{R} \rightarrow AC(\mathbb{R}_+)$ .

We assume further that there exists  $H_\circ : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that for every  $u_\Phi \in AC(\mathbb{R}_+)$  and for every  $y_{\Phi_0} \in \mathbb{R}$ , (4)

holds, where  $y_\Phi = \Phi(u_\Phi, y_{\Phi_0})$ . Such class of Duhem model is investigated in our previous results [15]. In particular, the results in [15] give the sufficient conditions for the existence of such  $H_\circ$  based only on the data of  $f_1$  and  $f_2$ .

An example of the hysteresis models which has CW I/O behavior is the Dahl model. The Dahl model [5], [13] is commonly used in mechanical systems, which represents the friction force with respect to the relative displacement between two surfaces in contact. The general representation of the Dahl model is given by

$$\begin{aligned} \dot{y}_\Phi(t) &= \rho \left| 1 - \frac{y_\Phi(t)}{F_c} \operatorname{sgn}(\dot{u}_\Phi(t)) \right|^r \\ &\quad \operatorname{sgn} \left( 1 - \frac{y_\Phi(t)}{F_c} \operatorname{sgn}(\dot{u}_\Phi(t)) \right) \dot{u}_\Phi(t), \end{aligned} \quad (7)$$

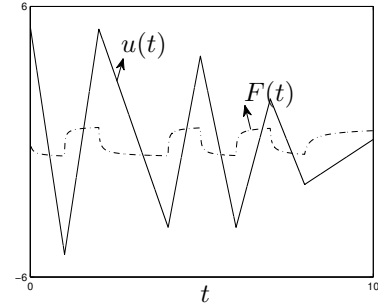
where  $y_\Phi$  denotes the friction force,  $u_\Phi$  denotes the relative displacement,  $F_c > 0$  denotes the Coulomb friction force,  $\rho > 0$  denotes the rest stiffness and  $r \geq 1$  is a parameter that determines the shape of the hysteresis loops.

The Dahl model can be described by the Duhem hysteresis operator (5) with

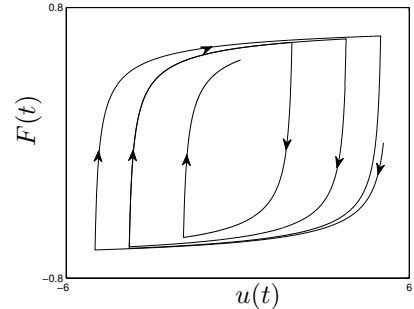
$$f_1(\nu, \xi) = \rho \left| 1 - \frac{\nu}{F_c} \right|^r \operatorname{sgn} \left( 1 - \frac{\nu}{F_c} \right), \quad (8)$$

$$f_2(\nu, \xi) = \rho \left| 1 + \frac{\nu}{F_c} \right|^r \operatorname{sgn} \left( 1 + \frac{\nu}{F_c} \right). \quad (9)$$

In Figure 2, we illustrate the behavior of the Dahl model where  $F_c = 0.75$ ,  $\rho = 1.5$  and  $r = 3$ .



(a)



(b)

Fig. 2. The input-output dynamics of the Dahl model with  $F_c = 0.75$ ,  $\rho = 1.5$  and  $r = 3$ .

Now let us consider a simple case when  $r = 1$ , the functions (8) and (9) become

$$f_1(\nu, \xi) = \rho \left( 1 - \frac{\nu}{F_c} \right), \quad f_2(\nu, \xi) = \rho \left( 1 + \frac{\nu}{F_c} \right). \quad (10)$$

It is immediate to check that the conditions as given in (6) are satisfied, which means there exists solution for this Duhem operator for all positive time.

Following the same construction procedure as in [15], we can compute explicitly  $H_\circ$  as follows

$$H_\circ(y_\Phi(t), u_\Phi(t)) = \begin{cases} \frac{F_c^2}{\rho} \ln \frac{F_c}{y_\Phi(t) + F_c} + \frac{F_c}{\rho} y_\Phi(t) & y_\Phi(t) \geq 0 \\ \frac{F_c^2}{\rho} \ln \frac{-F_c}{y_\Phi(t) - F_c} - \frac{F_c}{\rho} y_\Phi(t) & y_\Phi(t) < 0. \end{cases} \quad (11)$$

If  $-F_c < y_{\Phi_0} < F_c$ , then  $H_\circ$  satisfy (4).

### III. MAIN RESULTS

Consider a feedback interconnection of a second-order single-input single-output (SISO) linear system with a PID and a hysteretic actuator as shown in Figure 1, where  $\mathbf{P}$  represents the linear plant,  $\mathbf{C}$  is the PID controller and  $\Phi$  represents the Duhem hysteresis operator. In Figure 1, the measurement noise is represented by the disturbance  $d_1$  and  $d_2$ . The disturbance  $d_1$  and  $d_2$  can also be regarded as the estimation error which comes from the application of state observer for the plant  $\mathbf{P}$ . Later, in our simulation results, we simulate the robustness property of PID with respect to  $\dot{d}_1$  and  $\dot{d}_2$  by incorporating linear state observer for  $\mathbf{P}$  where  $u$  and  $x_2$  are not available for measurements.

The closed-loop system as shown in Figure 1, are given by

$$\begin{aligned} \mathbf{P}: \quad \dot{x} &= Ax + Bu, \\ \mathbf{C}: \quad \dot{z} &= x_1, \\ y_c &= k_i z + \begin{bmatrix} k_p & k_d \end{bmatrix} x - k_p d_1 - k_d d_2, \\ \Phi: \quad \dot{y}_\Phi &= f_1(y_\Phi, u_\Phi) \dot{u}_{\Phi+} + f_2(y_\Phi, u_\Phi) \dot{u}_{\Phi-}, \\ u_\Phi &= -y_c, \quad u = y_\Phi, \end{aligned} \quad (12)$$

where  $A \in \mathbb{R}^{2 \times 2}$ ,  $B \in \mathbb{R}^{2 \times 1}$ ,  $C \in \mathbb{R}^{1 \times 2}$  and  $x = [x_1 \ x_2]^T$ . For the PID controller  $\mathbf{C}$ ,  $z$  is the state of the integrator,  $k_p > 0$ ,  $k_i > 0$  and  $k_d > 0$  are the controller gains. For the Duhem hysteresis operator  $\Phi$ , the functions  $f_1$  and  $f_2$  are assumed to be locally Lipschitz.

**Theorem 3.1:** Consider system (12) with  $d_1 = 0$  and  $d_2 = 0$ . Assume that for the Duhem hysteresis operator  $\Phi$  there exists  $H_\circ : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that for every  $u_\Phi \in AC(\mathbb{R}_+)$  and for every admissible  $y_{\Phi_0}$ , (4) holds, where  $y_\Phi = \Phi(u_\Phi, y_{\Phi_0})$ . Suppose that either (i) the function  $H_\circ(\cdot, \xi)$  is proper (or radially unbounded) for every  $\xi$ ; or (ii) the solution  $y_\Phi = \Phi(u_\Phi, y_{\Phi_0})$  is bounded for every  $u_\Phi \in AC(\mathbb{R}_+)$  and every admissible  $y_{\Phi_0}$ . If there exists  $Q = Q^T > 0$ ,  $L$ ,  $k_p > 0$ ,  $k_i > 0$  and  $k_d > 0$  such that the following inequalities

$$A^T Q + QA + L^T L \leq 0 \quad (13)$$

$$QB - A^T \begin{bmatrix} k_p & k_d \end{bmatrix}^T - \begin{bmatrix} k_i \\ 0 \end{bmatrix} = L^T w \quad (14)$$

$$2 \begin{bmatrix} k_p & k_d \end{bmatrix} B = w^2 + v \quad (15)$$

hold for some  $w \in \mathbb{R}$ ,  $v > 0$  and the tuple  $(A, L)$  is observable, then for every initial conditions  $(x_0, z_0) \in \mathbb{R}^3$  and every admissible  $y_{\Phi_0} \in \mathbb{R}$ , the state trajectories of the closed-loop system (12) are bounded and converge to  $\mathcal{A} := \{(x, z, y_\Phi) \in \mathbb{R}^4 | x = 0, y_\Phi = 0\}$ , i.e. the closed-loop system is globally asymptotic stable with respect to  $\mathcal{A}$  ( $\mathcal{A}$ -GAS).

Note that  $v > 0$  in (15) will be used later to show the dissipativity property for robustness analysis.

*Proof:* By the assumption of the theorem, the Duhem operator  $\Phi$  is CW and there exists a function  $H_\circ : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that

$$\dot{H}_\circ(y_\Phi(t), u_\Phi(t)) \leq y_\Phi(t) \dot{u}_\Phi(t). \quad (16)$$

Using  $V_Q(x) = \frac{1}{2} x^T Q x$  and (13)-(15), we have

$$\begin{aligned} \dot{V}_Q(x) &= \frac{1}{2} x^T (A^T Q + QA) x + x^T Q B u \\ &= \dot{y}_c u + \frac{1}{2} x^T (A^T Q + QA) x + x^T Q B u - \dot{y}_c u \\ &= \dot{y}_c u + \frac{1}{2} x^T (A^T Q + QA) x + x^T \left( QB - \begin{bmatrix} k_i \\ 0 \end{bmatrix} \right) u \\ &\quad - x^T A^T \begin{bmatrix} k_p & k_d \end{bmatrix}^T u - \begin{bmatrix} k_p & k_d \end{bmatrix} B u^2, \\ &\leq \dot{y}_c u - \frac{1}{2} x^T L^T L x + x^T L^T w u - \frac{1}{2} w^2 u^2 - \frac{1}{2} v u^2, \\ &\leq \dot{y}_c u - \frac{1}{2} (Lx - wu)^T (Lx - wu) - \frac{1}{2} v u^2. \end{aligned}$$

Based on Definition 2.1, it can be easily checked that the linear system is CCW.

Now using  $V_{cl}(x, z, y_\Phi) = V_Q(x) + H_\circ(y_\Phi, -k_i z - \begin{bmatrix} k_p & k_d \end{bmatrix} x)$  as the Lyapunov function of the closed-loop system (12) and substituting the interconnection conditions  $u_\Phi = -y_c$  and  $u = y_\Phi$ , we obtain

$$\begin{aligned} \dot{V}_{cl} &= \dot{V}_Q + \dot{H}_\circ, \\ &\leq \dot{y}_c u - \frac{1}{2} (Lx - wu)^T (Lx - wu) + y_\Phi \dot{u}_\Phi, \\ &= -\frac{1}{2} (Lx - wu)^T (Lx - wu) - \frac{1}{2} v u^2. \end{aligned}$$

Since  $V_Q$  is proper in  $x$ , this inequality implies that  $x$  is bounded. Moreover, using assumption (i) together with the above inequality or using assumption (ii), we also have the boundedness of  $y_\Phi$ . Based on the LaSalle's invariance principle and the compactness of  $(x, y_\Phi)$ , the trajectories converge to the largest invariant set  $\mathcal{O}$  contained in  $\mathcal{M} := \{(x, y_\Phi) \in \mathbb{R}^3 | Lx - w y_\Phi = 0, y_\Phi = 0\}$  as  $t \rightarrow \infty$ .

Since  $y_\Phi = 0$  and  $Lx = 0$  in the invariant set  $\mathcal{O}$  in  $\mathcal{M}$ , the linear system  $\mathbf{P}$  (in the invariant set  $\mathcal{O}$ ) satisfies  $L A x = 0$ . By the observability of  $(A, L)$ ,  $Lx = 0$  and  $L A x = 0$  imply that  $x = 0$ . Thus, it follows that  $y_\Phi = 0$  and  $x = 0$  in  $\mathcal{O}$  and we conclude that (12) is GAS with respect to  $\mathcal{A} := \{(x, z, y_\Phi) \in \mathbb{R}^4 | x = 0, y_\Phi = 0\}$ .  $\square$

To illustrate Theorem 3.1, consider the following force-actuated mass-damper-spring system with PID controller and

a hysteretic actuator

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ \dot{z} &= x_1 \\ y_c &= k_i z + k_p x_1 + k_d x_2 \\ y_\Phi &= \Phi(u_\Phi) \\ u_\Phi &= -y_c, u = y_\Phi \end{aligned} \quad (17)$$

where  $x_1$  is the displacement,  $x_2$  is the velocity,  $z$  is the state of the integrator,  $k > 0$  is the spring constant,  $b > 0$  is the damping constant and  $m > 0$  is the mass.

**Corollary 3.2:** Assume that for the Duhem hysteresis operator  $\Phi$  there exists  $H_\circ : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that for every  $u_\Phi \in AC(\mathbb{R}_+)$  and for every admissible  $y_{\Phi_0} \in \mathbb{R}$ , (4) holds with  $y_\Phi := \Phi(u_\Phi, y_{\Phi_0})$ . If  $k_i \geq \frac{k k_d}{2m}$  and  $k_p \geq \frac{m k_i}{b} + \frac{b}{2m} k_d$ , then the closed-loop system (17) satisfies (13), (14) and (15) with  $L = [k\sqrt{\frac{k_d}{m}} \quad b\sqrt{\frac{k_d}{m}}]$ ,  $w = \sqrt{\frac{k_d}{m}}$ ,  $v = \frac{k_d}{m}$  and

$$Q = \begin{bmatrix} b k_i + k k_p - \frac{b k}{m} k_d & m k_i \\ m k_i & m k_p \end{bmatrix}.$$

In other words, (17) is  $\mathcal{A}$ -GAS.

In the next case, we study the robustness of the closed-loop system (12) by adding disturbances  $d_1$  and  $d_2$  to the measurement of the plant's state. This is related to the case where  $d_1$  and  $d_2$  are regarded as the state estimation error due to the application of state observer.

In [17], [2], a notion of *integral* input-to-state stability is introduced for the stability analysis of nonlinear systems given a bounded-energy input signal. It is shown in [2] that it is integral input-to-state stable (iISS) if the system is (a) 0-GAS and (b) dissipative with supply function  $\sigma$ . In [7] it is shown that for a class of dissipative systems, the iISS gain is equal to the supply function  $\sigma$ . Here, we explore the robustness of system (12) by applying the modified concept of iISS, where instead of discussing the iISS with respect to the origin, we are interested in the iISS property with respect to a set  $\mathcal{A}$  ( $\mathcal{A}$ -iISS). The definition of  $\mathcal{A}$ -iISS is given as follows.

**Definition 3.3:** Consider a system  $\dot{\zeta} = f(\zeta, u)$ ,  $\zeta(0) = \zeta_0$ , where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is locally Lipschitz. Let  $\mathcal{A} \subset \mathbb{R}^n$  be a nonempty and closed set. It is said to be *integral* input-to-state stable with respect to  $\mathcal{A}$  ( $\mathcal{A}$ -iISS) if there exist functions  $\alpha \in \mathcal{K}_\infty$ ,  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that, for every  $\zeta_0 \in \mathbb{R}^n$  and  $u \in L^\infty(\mathbb{R}^m)$ , the unique maximal solution  $\zeta$  is global and

$$\alpha(\|\zeta(t)\|_{\mathcal{A}}) \leq \beta(\|\zeta_0\|_{\mathcal{A}}, t) + \int_0^t \gamma(\|u(s)\|) ds \quad \forall t \in \mathbb{R}_+, \quad (18)$$

where  $\gamma$  is referred to as  $\mathcal{A}$ -iISS gain.

Before we discuss the robustness of (12), we precede it with the dissipativity property of (12).

**Lemma 3.4:** Consider the system in (12) with  $d_1, d_2 \in C^1(\mathbb{R}_+)$  s.t.  $\dot{d}_1, \dot{d}_2 \in AC(\mathbb{R}_+)$ . Assume that the hypotheses in Theorem 3.1 hold. Then there exists

$\mu > 0$ , such that (12) is dissipative with the supply function  $\sigma(\dot{d}_1, \dot{d}_2) = \mu \left\| \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \right\|^2$ .

*Proof:* Let  $V_{cl} = V_Q + H_\circ$  be the Lyapunov function for the closed-loop system, where  $V_Q$  and  $H_\circ$  have the same descriptions as in the proof of Theorem 3.1. Then we have

$$\begin{aligned} \dot{V}_{cl} &= \dot{V}_Q + \dot{H}_\circ, \\ &\leq \dot{y}_c u - \frac{1}{2}(Lx - wu)^T(Lx - wu) + y_\Phi \dot{u}_\Phi - \frac{1}{2}v u^2, \\ &= -\frac{1}{2}(Lx - wu)^T(Lx - wu) + k_p y_\Phi \dot{d}_1 + k_d y_\Phi \dot{d}_2 \\ &\quad - \frac{1}{2}v u^2 \end{aligned}$$

where the last equality is obtained since  $u_\Phi = -y_c + k_p d_1 + k_d d_2$ . Using Young's inequality and  $u = y_\Phi$ , we have

$$\begin{aligned} \dot{V}_{cl} &\leq \frac{k_d \epsilon}{2} \dot{d}_2^2 + \frac{1}{2\epsilon} y_\Phi^2 + \frac{k_p \delta}{2} \dot{d}_1^2 + \frac{1}{2\delta} y_\Phi^2 - \frac{1}{2}v y_\Phi^2, \\ &= \frac{k_p \delta}{2} \dot{d}_1^2 + \frac{k_d \epsilon}{2} \dot{d}_2^2 - \frac{1}{2}\left(v - \frac{1}{\epsilon} - \frac{1}{\delta}\right) y_\Phi^2 \end{aligned}$$

where  $\epsilon$  and  $\delta$  are arbitrary positive constants. Since  $v > 0$ , we can take  $\epsilon$  and  $\delta$  such that

$$\frac{1}{\epsilon} + \frac{1}{\delta} \leq v \quad (19)$$

holds. This implies that system (12) is dissipative with respect to  $\sigma$  where  $\sigma(\dot{d}_1, \dot{d}_2) = \mu \left\| \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \right\|^2$  and  $\mu = \max\{\frac{k_d \delta}{2}, \frac{k_d \epsilon}{2}\}$ .  $\square$

**Theorem 3.5:** Consider the system in (12) with  $d_1, d_2 \in C^1(\mathbb{R}_+)$  s.t.  $\dot{d}_1, \dot{d}_2 \in AC(\mathbb{R}_+)$ . Assume that the hypotheses in Theorem 3.1 hold and additionally, (19) is satisfied with  $\epsilon, \delta > 0$ . Assume that the functions  $f_1$  and  $f_2$  of the Duhem operator  $\Phi$  satisfy

**(B)** For every compact set  $K \in \mathbb{R}$ , there exists  $c > 0$  such that

$$\|f_1(\nu, \xi)\| \leq c, \quad \|f_2(\nu, \xi)\| \leq c \quad \forall \nu \in K, \xi \in \mathbb{R}. \quad (20)$$

Then (12) is iISS with respect to  $\mathcal{A}$  ( $\mathcal{A}$ -iISS), with iISS gain  $\gamma\left(\left\| \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \right\|\right) = \mu \left\| \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \right\|^2$ , where  $\mu > 0$ .

*Proof:* The proof of Theorem 3.5 follows the same arguments as in the proof of Theorem 3.1 in [7]. Firstly, the arguments in [17] which use the converse Lyapunov theorem for GAS system are replaced by the similar arguments using the converse Lyapunov theorem for  $\mathcal{A}$ -GAS system as discussed in [18].

Notice that the system (12) with  $d_1 = 0$  and  $d_2 = 0$  can be written explicitly as

$$\begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{y}_\Phi \end{bmatrix} = \begin{bmatrix} Ax + B y_\Phi \\ f_1(y_\Phi, u_\Phi)(-[k_i \quad 0] + [k_p \quad k_d]A)x - [k_p \quad k_d]B y_\Phi + \\ + f_2(y_\Phi, u_\Phi)(-[k_i \quad 0] + [k_p \quad k_d]A)x - [k_p \quad k_d]B y_\Phi \end{bmatrix}, \quad (21)$$

where  $u_\Phi = -k_i z - [k_p \quad k_d]x$ . It can be checked that the RHS of equation (21) is locally Lipschitz. Let us write (21) by  $\dot{\zeta} = f(\zeta)$  where  $\zeta = [x^T \ z \ y_\Phi]^T$ . Based on the converse

Lyapunov theorem [18, Corollary 2], Theorem 3.1 implies that there exists a smooth Lyapunov function  $V : \mathbb{R}^4 \rightarrow \mathbb{R}_+$  such that

- there exist  $\mathcal{K}_\infty$  functions  $\alpha_1$  and  $\alpha_2$  such that

$$\alpha_1(\|\zeta\|_{\mathcal{A}}) \leq V(\zeta) \leq \alpha_2(\|\zeta\|_{\mathcal{A}}) \quad \forall \zeta \in \mathbb{R}^4.$$

- there exists a continuous, positive definite function  $\alpha_3$  such that

$$\frac{dV(\zeta)}{d\zeta} f(\zeta) \leq -\alpha_3(\|\zeta\|_{\mathcal{A}}) \quad \forall \zeta \in \mathbb{R}^4.$$

With the disturbance  $d_1$  and  $d_2$ , the system (12) can be written as  $\dot{\zeta} = f(\zeta, d_1, d_2, \dot{d}_1, \dot{d}_2)$  where

$$f(\zeta, d_1, d_2, \dot{d}_1, \dot{d}_2) = \begin{bmatrix} x_1 \\ Ax + By_\Phi \\ f_1(y_\Phi, u_\Phi) - ([k_i \ 0] + [k_p \ k_d]A)x - [k_p \ k_d]By_\Phi - [k_p \ k_d] \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \\ + f_2(y_\Phi, u_\Phi) - ([k_i \ 0] + [k_p \ k_d]A)x - [k_p \ k_d]By_\Phi - [k_p \ k_d] \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \end{bmatrix},$$

$u_\Phi = -k_i z - [k_p \ k_d]x - [k_p \ k_d] \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix}$ . Using the Assumption (B), we have that for every  $\mathbb{B}_l^{\mathcal{A}} = \left\{ \begin{bmatrix} z \\ x \\ y_\Phi \end{bmatrix} \left\| \begin{bmatrix} x \\ y_\Phi \end{bmatrix} \right\| \leq l \right\}$ , there exists  $c > 0$  such that

$$\|f(\zeta, d_1, d_2, \dot{d}_1, \dot{d}_2)\| \leq c \left( 1 + \mu \left\| \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} \right\|^2 \right) \quad \forall (\zeta, d_1, d_2, \dot{d}_1, \dot{d}_2) \in \mathbb{B}_l^{\mathcal{A}} \times \mathbb{R}^4. \quad (22)$$

By replacing the compact set  $K \subset \mathbb{R}^n$  in [7, Assumption (A)] and in the rest of the proof in [7] by  $\mathbb{B}_l^{\mathcal{A}}$ , we can obtain the same lemmas as given in [7, Lemma 3.2, Lemma 3.3 and Lemma 3.4]. This together with the converse Lyapunov function for  $\mathcal{A}$ -GAS system, we can obtain the  $\mathcal{A}$ -iISS Lyapunov function similar to the proof of [7, Theorem 3.1].  $\square$

We remark that the Dahl model with  $f_1$  and  $f_2$  given by (8) and (9), respectively, satisfies the Assumption (B) in Theorem 3.5.

#### IV. SIMULATION RESULTS

In this section, we simulate the force-actuated mass-damper-spring system as given in (17) in Matlab. In the simulation we set  $m = 1$ ,  $k = 1$  and  $b = 2$ . The friction force is modeled by the Dahl hysteresis model as described in Section II-B with  $F_c = 0.75$ ,  $\rho = 1.5$  and  $r = 1$ . Then (17) becomes

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \dot{z} &= x_1 \\ y_c &= k_i z + k_p x_1 + k_d x_2 \\ \dot{y}_\Phi &= 1.5 \left( 1 - \frac{y_\Phi}{0.75} \right) \dot{u}_{\Phi+} + 1.5 \left( 1 + \frac{y_\Phi}{0.75} \right) \dot{u}_{\Phi-} \\ u_\Phi &= -y_c, u = y_\Phi \end{aligned} \quad (23)$$

The control parameters are chosen as  $k_p = 1$ ,  $k_i = 1/2$  and  $k_d = 1/4$ . It can be easily checked that by taking  $Q = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ , the inequalities (13)-(15) are satisfied.

First, let us consider the case when there is no disturbance in the system. Given the initial condition  $x_{10} = -5$ ,  $x_{20} = 10$  and  $y_{\Phi 0} = 0.5$ , Figure 3 shows that the state trajectories converge to the invariant set  $\mathcal{A} := \{(x_1, x_2, z, y_\Phi) | x_1 = 0, x_2 = 0, y_\Phi = 0\}$  in agreement with Theorem 3.1.

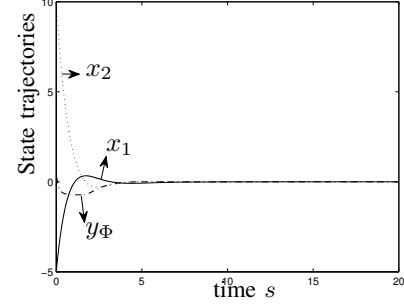


Fig. 3. The state trajectories of the closed-loop system (23) with  $x_{10} = -5$ ,  $x_{20} = 10$  and  $y_{\Phi 0} = 0.5$ .

For the next case, we add the disturbance  $d_1 = e^{-t}$  and  $d_2 = e^{-2t}$  as in (12). The simulation results are shown in Figure 4, which indicate that all the state trajectories are still converging to  $\mathcal{A}$ . This is a consequence of the  $\mathcal{A}$ -iISS property of the closed-loop system as presented in Theorem 3.5.

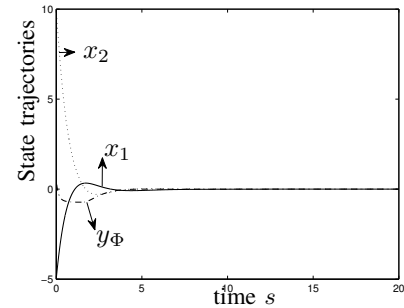


Fig. 4. The state trajectories of the closed-loop system (23) with  $x_{10} = -5$ ,  $x_{20} = 10$ ,  $y_{\Phi 0} = 0.5$ ,  $d_1 = e^{-t}$  and  $d_2 = e^{-2t}$ .

As an example of the implementation of state observer in the closed-loop system, we assume that the velocity of the mass-damper-spring system cannot be measured, which means that the information of the velocity needs to be obtained via an observer. Here we design a linear observer for the plant, which has the following form

$$\dot{\hat{x}} = A\hat{x} + L_o(y - \hat{y}) \quad (24)$$

where  $\hat{x}$  denotes the states of the observer,  $L_o$  is the observer gain,  $y$  is the output of the linear plant and  $\hat{y}$  is the output of the observer. Note that the input  $u$  is absent in (24) since it can not be measured from the output of the

hysteretic actuator. By considering  $u$  as the disturbance to the observer error dynamics, the design of the observer amounts to ensuring that the small-gain condition between the gain of the observer error dynamics and the  $i$ ISS gain of (12) is fulfilled in the spirit of  $i$ ISS domination [19, Proposition 1].

Due to the limited space, we show only the simulation result where we design an observer with a sufficiently fast convergence rate (so that it has a very small gain from  $u$  to the estimation error  $\begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} - \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix}$ ). Using  $L_o = [10 \ 4]^T$ ,  $x_0 = [10 \ -5]^T$  and  $\hat{x}_0 = [-10 \ 5]$ , we show in Figure 5 that the observer state, the plant state and  $y_\Phi$  converge to zero as expected.

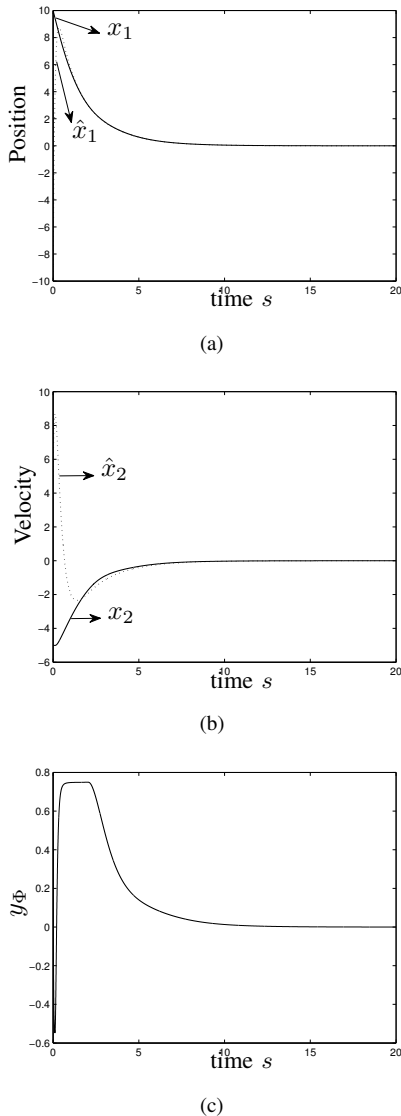


Fig. 5. The state trajectories of the closed-loop system (23) with  $x_0 = [10 \ -5]^T$ ,  $\hat{x}_0 = [-10 \ 5]$  and  $y_{\Phi 0} = 0.5$ .

## V. CONCLUSION

In this work we study the stability of a second order linear system with CW Duhem hysteresis, where we use the standard proportional, integral and derivative (PID) controller. In

particular, we give sufficient conditions on the linear system and control parameters such that the origin of the plant and the hysteresis state is attractive. Furthermore, for the robustness analysis, we show that the closed-loop system is  $\mathcal{A}$ - $i$ ISS with respect to the measurement noise in the closed-loop system.

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